Indian Statistical Institute, Bangalore
B. Math. Third Year
Second Semester - Analysis IV

Backpaper Exam | Duration: 3 hours | Date : May 25, 2015

Each question carries 10 marks

1. Let X be a compact metric space and $\mathcal{A} \subset C_{\mathbb{R}}(X)$ be a subalgebra that separates points of X and nowhere vanishes on X. Prove that $\mathcal{A}$ is dense in $C_{\mathbb{R}}(X)$.

2. (a) Prove that $C[0,1]$ has no open set whose closure is compact (**Marks: 4**).

   (b) Let X be a complete metric space and $\phi\colon X \to X$ be a map such that $d(\phi^n(x), \phi^n(y)) \leq a_n d(x,y)$ for all $n \geq 1$ and all $x, y \in X$ for some sequence $a_n \to 0$. Prove that ϕ has a unique fixed point $x \in X$ and $\lim_{n\to\infty} \phi^n(y) = x$ for all $y \in X$.

3. (a) Let $E \subset \mathbb{R}^n$ be an open set and $f\colon E \to \mathbb{R}^n$ be a C^1-map with $f'(x)$ is invertible. Prove that there is neighborhood U of x such that $f(U)$ is open in $\mathbb{R}^n$.

   (b) Prove that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ (**Marks: 4**).

4. (a) Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $s_n(x)$ be the n-th partial sum of the Fourier series. If f is differentiable at x, prove or disprove $s_n(x) \to f(x)$.

   (b) Prove that $\sum_1^\infty \frac{1}{n^2} = \frac{\pi - x}{2}$ for $0 < x < \pi$(**Marks: 5**).

5. (a) Let $f \in \mathcal{R}[a.b]$. Prove that $\lim_{r\to\infty} \int_a^b f(t)\sin(rt+s)dt = 0$ for any $s \in \mathbb{R}$.

   (b) Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function and $f \sim \sum_{-\infty}^\infty c_n e^{inx}$. If $\sum n^2|c_n|^2 < \infty$, prove that $\sum_{-\infty}^\infty c_n e^{inx} = f(x)$ uniformly.